Mathematical modelling in science and engineering

Lecture 4
Finite element approximation and adaptivity

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Finite element approximation of elliptic problems

- Model elliptic problem in 1D computational domain $\Omega = (0, 1)$
  - differential equation:
    $$-\frac{d^2u}{dx^2} = f(x)$$
  - boundary conditions:
    - Dirichlet:
      $$u(0) = u_0$$
    - Neumann:
      $$\frac{du}{dx}(1) = u_1^{'}]$$

Weak formulation

Find a function $u(x) \in V$ such that the following holds:

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx = \int_0^1 f(x) \cdot w(x)dx + u_1^{'} \cdot w(1) \qquad \forall w \in V_0$$

Finite element approximation of elliptic problems

Abstract formulation of finite element problems

- For the simplest problems both sides in the weak formulations can be identified with scalar products
- For any problem its weak formulation can be stated as:

  Find $u \in V$ such that

$$a(u, w) = L(w) \qquad \forall w \in \tilde{V}$$

- Finite element formulations for spaces consisting of all functions being linear combinations of basis functions is then the following:

  Find $u_h \in V_h$ such that

$$a(u_h, w_h) = L(w_h) \qquad \forall w_h \in \tilde{V}_h$$

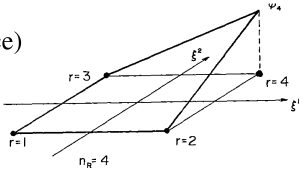
  where $a(.,.)$ is a bilinear form and $L(.)$ is a linear form, i.e. for example:

$$a(\alpha u_1 + \beta u_2, w) = \alpha a(u_1, w) + \beta a(u_2, w)$$

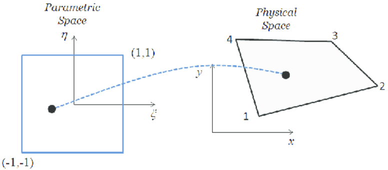
Finite element approximation of elliptic problems

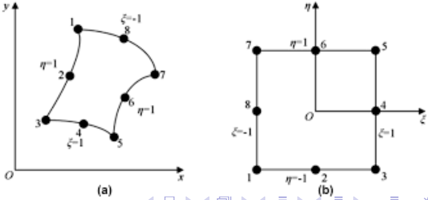
Definition of finite element approximation spaces V_h:

- element shape functions
  - most often polynomials (in 1D, 2D or 3D space)
  - defined on so called reference elements - simple domains like $[-1, 1]$ interval or $[0, 1] \times [0, 1]$ rectangle



- domain discretization into finite elements
- transformation of reference elements to real elements
  - translated, rotated, stretched - for linear transformations
  - possibly made curvilinear - for geometrically higher order elements (used e.g. to approximate circular domains etc.)

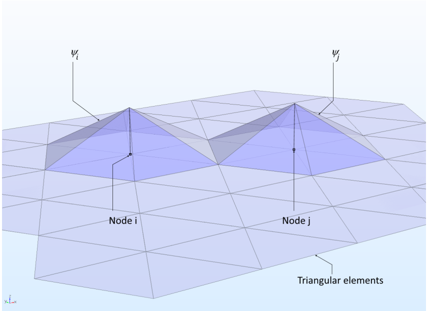


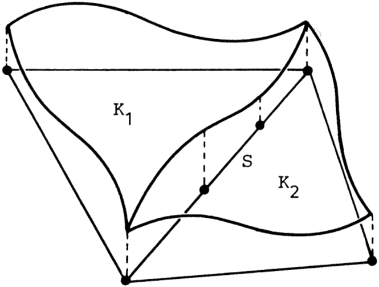


Finite element approximation of elliptic problems

Definition of finite element approximation spaces V_h:

- definition of basis functions based on shape functions for real elements
- prescription how to construct global basis functions from shape functions
    - usually glued together based on the notion of global finite element nodes (degrees of freedom) and the requirement of continuity
    - for some problems and formulations, spaces of discontinuous functions are introduced

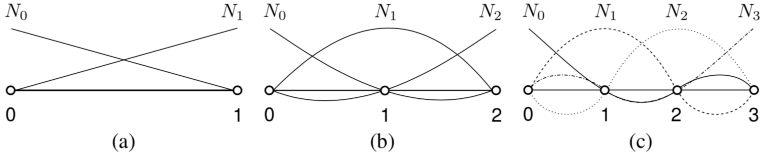




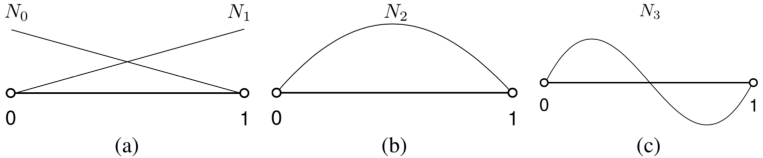
Finite element approximation of elliptic problems

Example approximation spaces in 1D

- Linear shape functions (lectures 3 and 4)
- Higher order shape functions
  - Lagrange polynomials



  - hierarchical polynomials

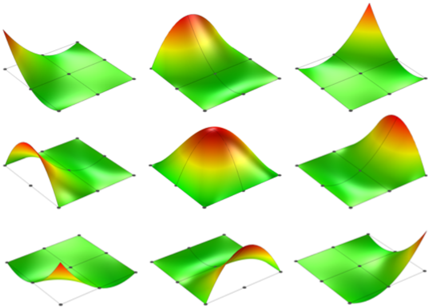


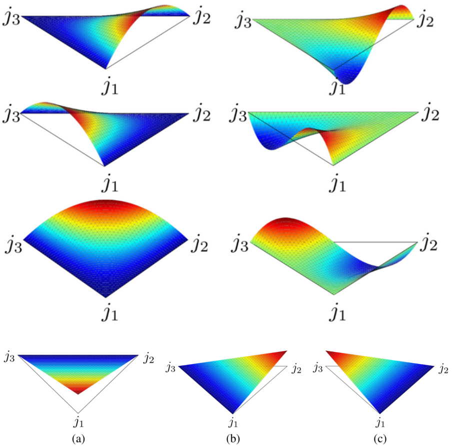
  - polynomials with higher smoothness: Hermite polynomials, splines etc.

Finite element approximation of elliptic problems

Example approximation spaces in 2D

- second order (quadratic)
- hierarchical for triangles ->
- Lagrange for quadrilaterals

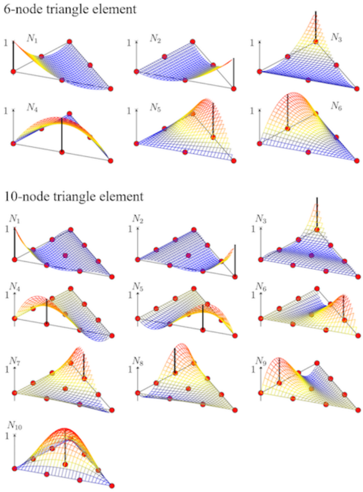




Finite element approximation of elliptic problems

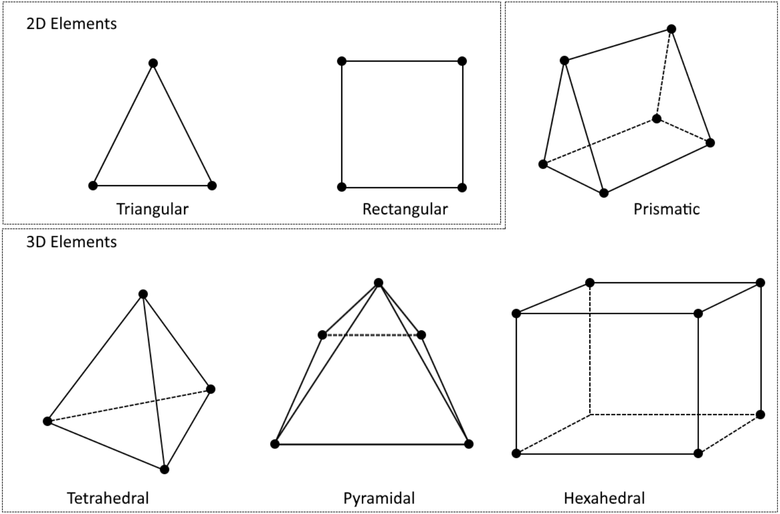
Properties of finite element approximation types

- Lagrange polynomials
  - related to finite element nodes – each basis function is equal to one at a single node and is equal 0 at all other nodes
  - degrees of freedom as values at finite element nodes
  - when changing the order of approximation all shape functions have to be redefined
- hierarchical polynomials
  - to increase the order of approximation it is sufficient to add additional shape functions



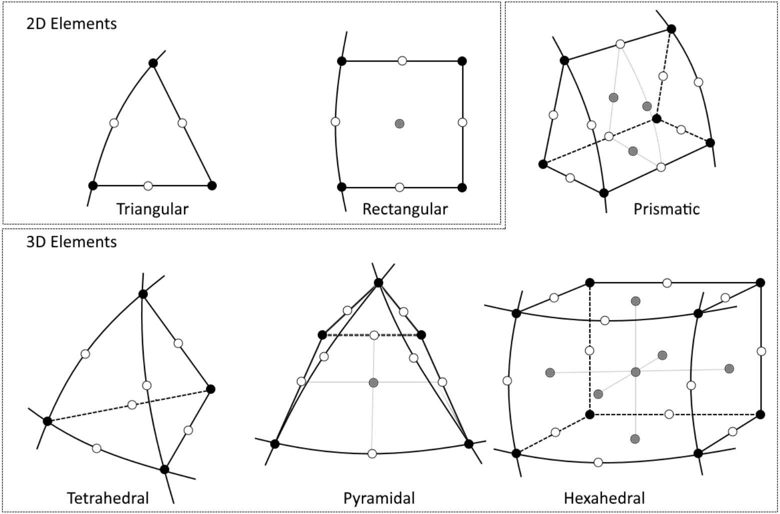
Finite element approximation of elliptic problems

Typical linear finite elements in 2D and 3D



Finite element approximation of elliptic problems

Typical quadratic Lagrange finite elements in 2D and 3D



Finite element approximation of elliptic problems

Norms for measuring error (and other functions defined over Ω)

- L_2

$$\|e\|^2_{L_2(\Omega)} = \int_\Omega (e \cdot e) d\Omega$$

- H^1

$$\|e\|^2_{H^1(\Omega)} = \int_\Omega \left(e_{,i} \cdot e_{,i} + e \cdot e\right) d\Omega$$

- H^1 seminorm

$$|e|^2_{H^1(\Omega)} = \int_\Omega (e_{,i} \cdot e_{,i}) d\Omega$$

- energy norm
  - for many problems their bilinear forms satisfy the requirements for scalar products and, hence, can be used to define a norm:

$$\|e\|^2_a = a(e,e)$$

Finite element approximation of elliptic problems

- Fundamental error estimate for finite element approximation of elliptic problems is

$$\|u - u_h\|_V \leq C \cdot \|u - w_h\|_V \qquad \forall w_h \in \tilde{V}_h$$

where $\|.\|_V$ is a norm induced by the scalar product defined for the space V
- The standard method of obtaining absolute error estimates for finite element approximation is to select a particular suitable function w_h (usually interpolant of u in V_h) and then obtaining error estimates for w_h
- Interpolation theory gives error estimates for interpolants in different finite element spaces – for linear, quadratic, etc. shape functions
- For standard continuous polynomials of order p one can finally get:

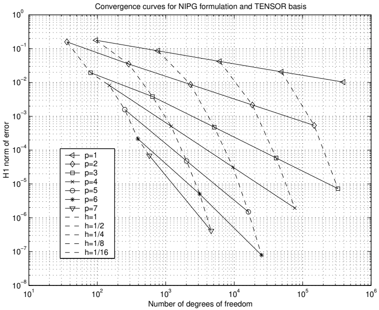
$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^p |u|_{H^{p+1}(\Omega)}$$

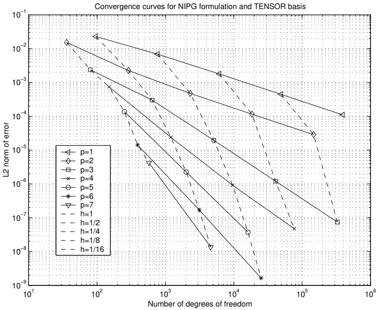
- The estimate requires the exact solution u to be sufficiently smooth, that depends on the problem and the shape of the computational domain Ω

Finite element approximation of elliptic problems

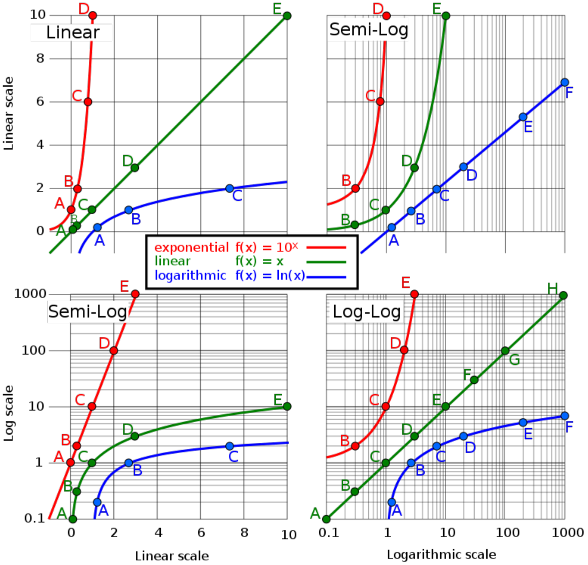
Typical convergence curves for finite element approximation of elliptic problems, measured in L_2 and H^1 norms

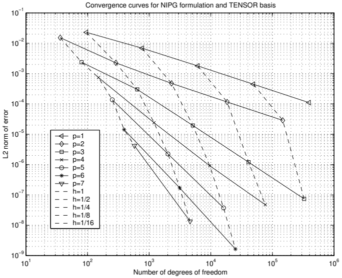
- log-log scale to explicitly show convergence rates
- the solution usually converges in L_2 norm with the rate h^{p+1}
- higher order approximations have better accuracy, but require more computational resources, for the same number of degrees of freedom





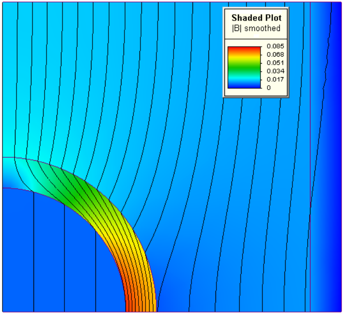
Intermezzo - how to read graphs

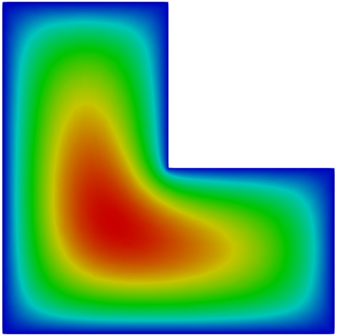




Finite element approximation of elliptic problems

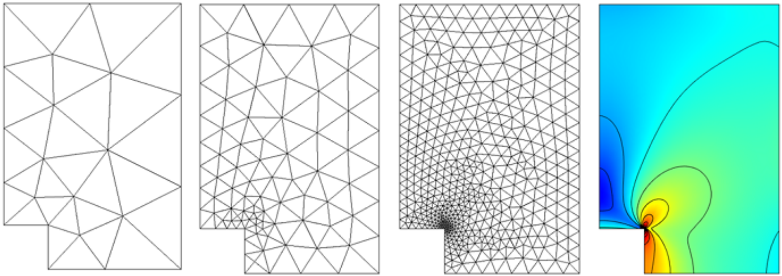
- The error of the finite element solution is related to the smoothness of the exact solution
  - for certain problems (e.g. with discontinuous coefficients - left) and for certain computational domains (e.g. with corners - right) the exact solution has large higher order derivatives
  - the convergence rates for such problems and uniform mesh refinements are slow





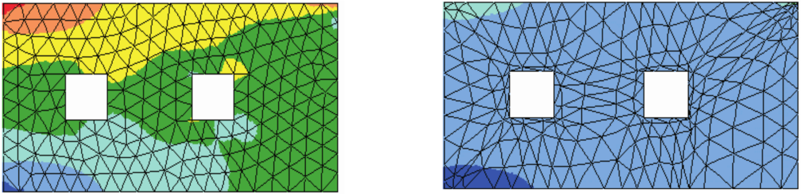
Finite element approximation of elliptic problems

- The nature of the finite element error estimates suggests that it is possible to decrease approximation error, especially for the problems with singularities, by local changes to approximation properties
- This observation gives rise to the adaptive finite element method, where in the places with higher approximation error the approximation is locally improved



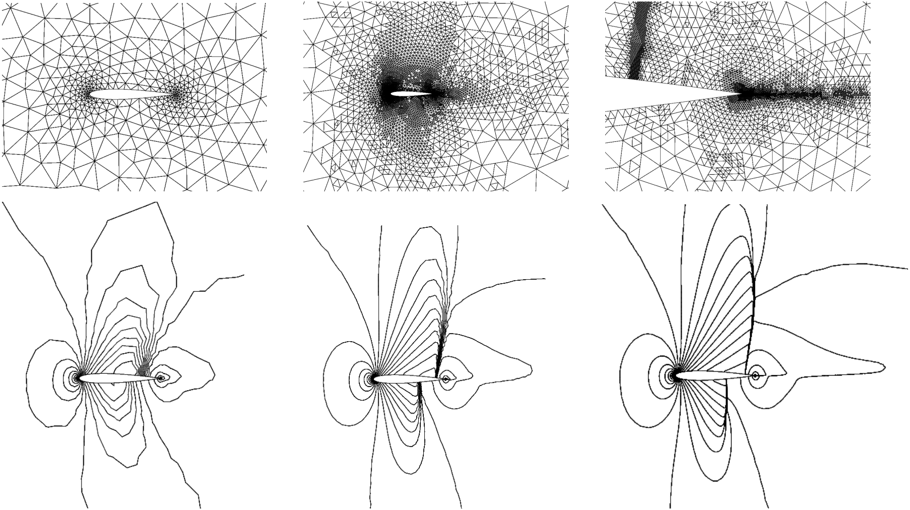
Finite element approximation of elliptic problems

- There are several main types of adaptivity:
  - *h*-adaptivity – the size of elements is reduced by dividing elements (the number of degrees of freedom grows)
  - *p* adaptivity – the local order of approximation is increased (this requires special techniques to maintain the continuity of the solution)
  - *hp*-adaptivity – the combination of the two above
  - *r*-adaptivity – the finite element nodes are moved, in order to create parts of the domain with smaller elements (the total number of degree of freedom may remain the same)
  - remeshing – creating a new, finer grid for the selected parts of the domain (or a new mesh with variable ”density” of finite element nodes)



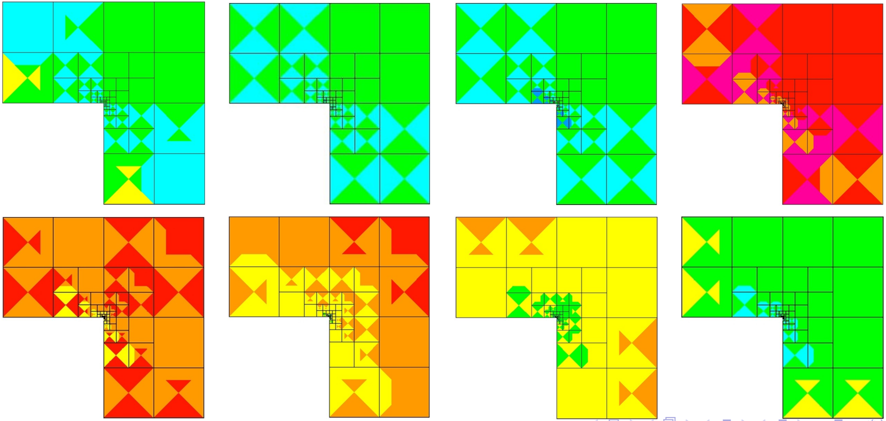
Finite element approximation of elliptic problems

- *h* adaptivity at work



Finite element approximation of elliptic problems

- *hp*-adaptive approximation for an elliptic problem in the L-shape domain
- subsequent figures show adapted meshes with increasing magnification, up to 100000000, colours represent the order of approximation p, from 1 (blue) to 6 (pink)



Finite element approximation of elliptic problems

- The justification for using *hp*-adaptivity is its best convergence rate
  - while standard (even higher order) *h*-adaptive FEM converges algebraically, *hp*-adaptive FEM has exponential convergence rates
- The main problems of *hp*-adaptivity are:
  - adaptive strategies – the selection which of the two options apply for a given element
  - complex coding
  - the limited number of problems for which *hp*-adaptivity can show its full potential